Analysis of Radiating Planar Resonators with the Method of Lines

A. Dreher and R. Pregla

FernUniversität, Postfach 940, W-5800 Hagen, Germany

Abstract

The method of lines (MOL) is extended to analyze radiating planar resonators by the use of absorbing boundary conditions. The complex frequency of a microstrip patch is computed and compared to results achieved by the integral equation method.

1 Introduction

The method of lines (MOL) is an efficient tool for the analysis of planar waveguide structures with multiple layers. In this paper it is extended to analyze radiating microstrip patches with various geometries. To limit the area of discretization, the structure is enclosed by walls, on which absorbing boundary conditions are used to simulate the free space [1]. The discretization is performed in two dimensions, while an analytical solution is used in the remaining direction. In this way, only four of these walls are necessary.

As an example, the complex resonant frequency of a rectangular microstrip patch (Fig. 1) is computed and compared with results obtained by the integral equation method.

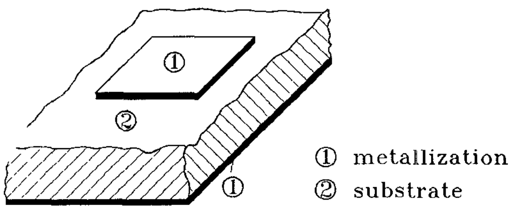

Figure 1: Simple Microstrip patch

2 Analysis

Assuming a time dependence $\exp(j\omega t)$ the wave equation, normalized by k_0, for the two independent field components e_z, h_z can be written as

$$L\psi = \left(D_{\bar{x}}^2 + D_{\bar{y}}^2 + D_{\bar{z}}^2 + \varepsilon_r\right)\psi = 0\,, \tag{1}$$

with

$$D_{\bar{x}}^2 \equiv \frac{\partial^2}{\partial \bar{x}^2}\,, \qquad D_{\bar{y}}^2 \equiv \frac{\partial^2}{\partial \bar{y}^2}\,, \qquad D_{\bar{z}}^2 \equiv \frac{\partial^2}{\partial \bar{z}^2}\,. \tag{2}$$

The Helmholtz operator L can be factored so that

$$L\psi = L^+ L^- \psi = 0\,, \tag{3}$$

from which follow the boundary conditions

$$L_x^{\pm}\psi = 0\,, \qquad L_z^{\pm}\psi = 0\,, \tag{4}$$

with

$$L_x^{\pm} = D_{\bar{x}}^2 \pm j\sqrt{\varepsilon_r}\sqrt{1 + s_{\bar{y}\bar{z}}^2}\,, \qquad s_{\bar{y}\bar{z}}^2 = \frac{1}{\varepsilon_r}\left(D_{\bar{y}}^2 + D_{\bar{z}}^2\right) \tag{5}$$

in x-direction and

$$L_z^{\pm} = D_{\bar{z}}^2 \pm j\sqrt{\varepsilon_r}\sqrt{1 + s_{\bar{x}\bar{y}}^2}\,, \qquad s_{\bar{x}\bar{y}}^2 = \frac{1}{\varepsilon_r}\left(D_{\bar{x}}^2 + D_{\bar{y}}^2\right) \tag{6}$$

in z-direction. The plus sign is related to waves traveling in the positive, the minus sign to those traveling in the negative direction of co-ordinates. For a unique solution the sign must be chosen properly at each wall so that only waves incident from the interior of the enclosed structure are admitted.

To use these boundary conditions in connection with the method of lines the radical is approximated by a polynomial of the form

$$\sqrt{1+s^2} \approx p_0 + p_2 s^2 \tag{7}$$

on the interval $s \in [-j, j]$. The advantageous procedure with shifted line systems to set up a system matrix can be performed in an elegant way in the method of lines by the use of dual boundary conditions [2]. For this reason, the equations (4) are applied to the tangential field components e_z, e_y and e_x, e_y respectively, resulting in

$$L_x'^{\pm} e_z = 0\,, \qquad L_x'^{\pm}\frac{\partial h_z}{\partial \bar{x}} = 0 \tag{8}$$

with

IF1

$$L'^{\pm}_x = D^2_{\bar{y}} + D^2_{\bar{z}} \mp j\frac{\sqrt{\varepsilon_r}}{p_{x2}} D_{\bar{z}} + \varepsilon_r \frac{p_{x0}}{p_{x2}} \tag{9}$$

on walls parallel to the z-axis and

$$L'^{\pm}_z \frac{\partial e_z}{\partial \bar{z}} = 0\,, \qquad L'^{\pm}_x h_z = 0 \tag{10}$$

with

$$L'^{\pm}_z = D^2_{\bar{y}} + D^2_{\bar{x}} \mp j\frac{\sqrt{\varepsilon_r}}{p_{z2}} D_{\bar{z}} + \varepsilon_r \frac{p_{z0}}{p_{z2}} \tag{11}$$

on walls parallel to the x-axis.

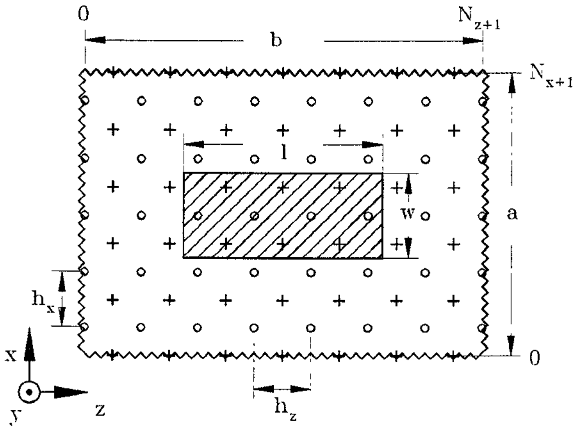

Figure 2: Microstrip patch with discretization lines and absorbing boundaries

Wave equation and boundary conditions are now discretized by a set of two line systems parallel to the y-axis (Fig. 2), and the resulting difference equation

$$\left(\hat{\boldsymbol{I}} D^2_{\bar{y}} - \widehat{\overline{\boldsymbol{P}}}_x - \widehat{\overline{\boldsymbol{P}}}_z + \varepsilon_r \hat{\boldsymbol{I}}\right) \boldsymbol{\Psi} = \boldsymbol{0} \tag{12}$$

is transformed by

$$\boldsymbol{\Psi} = \hat{\boldsymbol{T}} \overline{\boldsymbol{\Psi}} \tag{13}$$

to get a system of uncoupled ordinary differential equations

$$\left(\hat{\boldsymbol{I}} D^2_{\bar{y}} - \hat{\boldsymbol{k}}^2_{\bar{y}}\right) \overline{\boldsymbol{\Psi}} = 0\,. \tag{14}$$

$\hat{\boldsymbol{T}}$ is the transformation matrix for the two-dimensional case given by the Kronecker product

$$\hat{\boldsymbol{T}} = \boldsymbol{T}_z \otimes \boldsymbol{T}_x\,, \tag{15}$$

and

$$\hat{\boldsymbol{k}}^2_{\bar{y}} = \widehat{\overline{\boldsymbol{\lambda}}}^2_x + \widehat{\overline{\boldsymbol{\lambda}}}^2_z - \varepsilon_r \hat{\boldsymbol{I}}\,. \tag{16}$$

$\boldsymbol{T}_{x,z}$ are the matrices of the eigenvectors of

$$\boldsymbol{T}^{-1}_{x,z} \overline{\boldsymbol{P}}_{x,z} \boldsymbol{T}_{x,z} = \overline{\boldsymbol{\lambda}}^2_{x,z} \tag{17}$$

and the eigenvalues are given by

$$\widehat{\overline{\boldsymbol{\lambda}}}^2_x = \boldsymbol{I}_z \otimes \overline{\boldsymbol{\lambda}}^2_x\,, \qquad \widehat{\overline{\boldsymbol{\lambda}}}^2_z = \overline{\boldsymbol{\lambda}}^2_z \otimes \boldsymbol{I}_x\,. \tag{18}$$

The difference matrix

$$\overline{\boldsymbol{P}}_x = \bar{h}_x^{-2} \begin{bmatrix} p_{x11} & p_{x12} & p_{x13} & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & p_{x13} & p_{x12} & p_{x11} & \end{bmatrix}, \tag{19}$$

with

$$p_{x11} = 2 + a_x\,, \qquad p_{x12} = -1 - b_x\,, \qquad p_{x13} = 0 \tag{20}$$

for $\boldsymbol{\Psi} = \mathbf{E}_z$ or

$$p_{x11} = 1 + a_x\,, \qquad p_{x12} = -1 - b_x - a_x\,, \qquad p_{x13} = b_x \tag{21}$$

for $\boldsymbol{\Psi} = \mathbf{H}_z$, includes the boundary conditions by means of the coefficients

$$a_x = -2\frac{2p_{x2} + (p_{x0} - p_{x2})\, n_x^2}{2p_{x2} + jn_x}\,, \qquad b_x = -\frac{2p_{x2} - jn_x}{2p_{x2} + jn_x}\,, \tag{22}$$

with $n_x = \sqrt{\varepsilon_r}\,\bar{h}_x$. p_{x0} and p_{x2} are the coefficients of the approximation in (7). A similar representation can be found for the matrix $\overline{\boldsymbol{P}}_z$.

(14) is solved analytically and after some algebraic manipulation a cascade matrix is obtained for the tangential field components at the interfaces of an arbitrary dielectric layer $\imath$,

$$\begin{bmatrix} \overline{\mathbf{E}}_\imath \\ \overline{\mathbf{H}}_\imath \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{V}}_\imath & \overline{\boldsymbol{Z}}_\imath \\ \overline{\boldsymbol{Y}}_\imath & \overline{\boldsymbol{V}}_i \end{bmatrix} \begin{bmatrix} \overline{\mathbf{E}}_{\imath-1} \\ \overline{\mathbf{H}}_{i-1} \end{bmatrix} \tag{23}$$

with

$$\overline{\mathbf{H}}_\imath = \eta_0 \begin{bmatrix} \overline{\mathbf{H}}_{z\imath} \\ -\overline{\mathbf{H}}_{x\imath} \end{bmatrix}, \qquad \overline{\mathbf{E}}_\imath = j \begin{bmatrix} \overline{\mathbf{E}}_{x\imath} \\ \overline{\mathbf{E}}_{z\imath} \end{bmatrix}. \tag{24}$$

At the interface to the open top layer n the relation between the field components is

$$\overline{\mathbf{H}}_n = \overline{\boldsymbol{Y}}_n \overline{\mathbf{E}}_n\,, \tag{25}$$

in which $\overline{\boldsymbol{Y}}_n$ is a function of $\hat{\boldsymbol{k}}_{\bar{y}}$ and the sign of the radical, following from (16), must be chosen so that

$$\Im\left\{k_0 k_{\bar{y}_{\imath\jmath}}\right\} > 0 \tag{26}$$

to get an outgoing wave.

Starting with a short ($\overline{\mathbf{E}}_0 = \mathbf{0}$) for the metallization at the bottom of the structure, cascading the matrices of the several layers and taking into account the continuity equations

$$\overline{\mathbf{E}}^+_m = \overline{\mathbf{E}}^-_m\,, \qquad \overline{\mathbf{H}}^+_m - \overline{\mathbf{H}}^-_m = \eta_0 \begin{bmatrix} \overline{\mathbf{J}}_{xm} \\ \overline{\mathbf{J}}_{zm} \end{bmatrix}, \tag{27}$$

for the tangential field components on the lower (−) and the upper (+) side of the interfaces m with metallizations, the system equation

$$\overline{\boldsymbol{Z}} \cdot \overline{\boldsymbol{J}}_m = \overline{\boldsymbol{E}}_m \tag{28}$$

in the transform domain can be formulated. Since the tangential electric field components vanish on the metallizations, (28) is transformed back to spatial domain and the nonstandard eigenvalue problem

$$\boldsymbol{Z}_{\text{red}}\mathbf{J}_m = \mathbf{0} \tag{29}$$

with a reduced system matrix $\boldsymbol{Z}_{\text{red}}$ has to be solved for the complex resonant frequency ω.

3 Results

A single rectangular microstrip patch with one dielectric layer of thickness d (Fig. 2) has been investigated. The use of dual boundary conditions makes it possible, to insert electric or magnetic walls to utilize the symmetry of the structure and to reduce the amount of necessary computer storage. A Taylor series approximation, $p_0 = 1$, $p_2 = 1/2$, has been used for the operator on every wall. Computed results of the complex resonant frequency as a function of the resonator length are presented in Fig. 3a,b. They show a very good agreement to those achieved with the integral equation method [3].

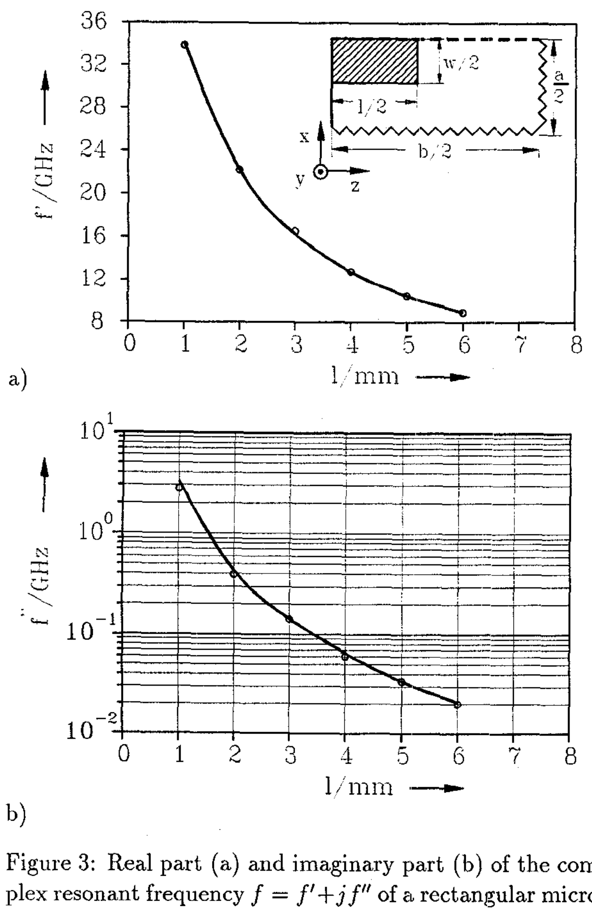

Figure 3: Real part (a) and imaginary part (b) of the complex resonant frequency $f = f' + jf''$ of a rectangular microstrip patch as a function of length l.
—— Method of lines, o Nam and Itoh [3]
($\varepsilon_{rc} = 9.6$, $w = 0.635$ mm, $d = 0.635$ mm. For good convergence, the distance between the walls, a, b, is more than one effective wavelength.)

In contrast to the integral equation method the application of the method of lines is simple, because no complicated analysis concerning branch cuts, surface wave poles and the course of the integration path has to be performed. Several

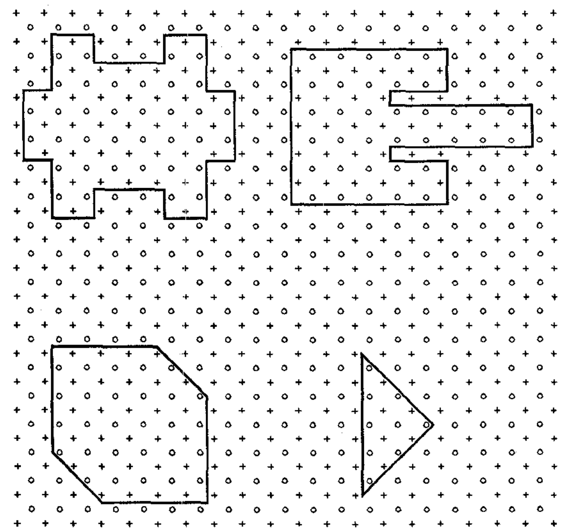

Figure 4: Examples of microstrip resonators and their discretization with the method of lines.

structures with various shapes and multiple arbitrary thin layers, as applied in hyperthermia and geophysics, can be analyzed (Fig. 4, 5). Optional metallizations at the in-

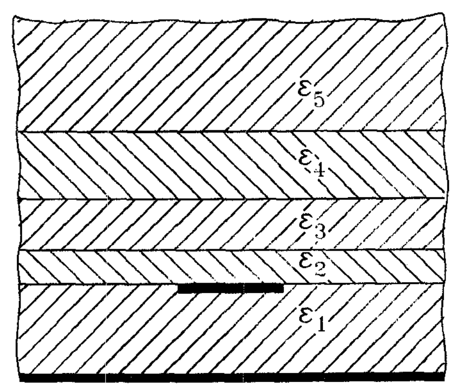

Figure 5: Resonator structures with multiple layers

terfaces (stacked patches) can be included and there is no difficulty to consider radiation effects in both vertical directions (Fig. 6). To this end, only an equation similar to (25) has to be taken into account at the interface to the bottom layer, whereas in the integral equation method an additional branch cut is necessary and more surface wave poles appear.

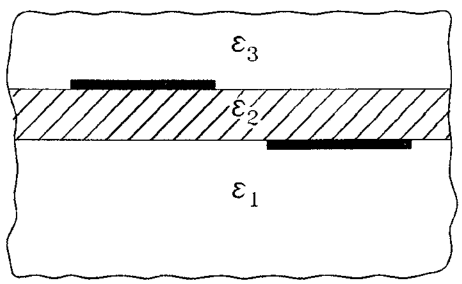

Figure 6: Radiation in two directions

References

[1] A. Dreher and R. Pregla, "'Analysis of Planar Waveguides with the Method of Lines and Absorbing Boundary Conditions"', *IEEE Microwave and Guided Wave Lett.,* vol. 1, pp. 138-140, June 1991.

[2] R. Pregla and W. Pascher, "The method of lines," in *Numerical Techniques for Microwave and Millimeter-Wave Passive Structures,* T. Itoh, Ed., New York: Wiley, 1989.

[3] S. Nam and T. Itoh, "'Calculation of Accurate Complex Resonant Frequency of an Open Microstrip Resonator using the Spectral Domain Method"', *J. Electromag. Waves Appl.,* 2, No. 9, S. 635-651, 1988.